10.1 Matrices and Systems of Equations

Date: __________

Matrices are used to solve systems of equations.

\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \]

Order of a Matrix: Rows X Columns

The order of the matrix above is: \( 3 \times 2 \)

Given Matrix \( A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & -3 \end{bmatrix} \)

a. What is the Order?

\( 2 \times 3 \)

b. What is the entry for \( a_{21} \)?

\( 0 \)

c. What entry is 1?

\( a_{1,3} \)

Elementary Row Operations

1. Interchange 2 rows

\[ R_1 \rightarrow R_2 \]

\[ \begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & -3 \end{bmatrix} \]

2. Multiply a row by a constant (not 0)

\[ -3R_1 \]

\[ \begin{bmatrix} 0 & -9 & -3 \\ 2 & 3 & 1 \end{bmatrix} \]

3. Add a multiple of a row to another

\[ 2R_1 + R_2 \]

\[ \begin{bmatrix} 2 \quad 10 \quad -1 \end{bmatrix} \]
Given \( B = \begin{bmatrix} 1 & 0 \\ -2 & 3 \\ 4 & 6 \end{bmatrix} \), find the elementary row operation performed.

1. \( R_1 \leftarrow R_2 \)
2. \( 2R_1 + R_2 \)
3. \( R_2 + R_3 \)
4. \( 3R_3 + R_1 \)

**Augmented Matrix:** A matrix derived from a system of equations.

System of Equations

\[
\begin{align*}
4x + 2y - 3z &= -1 \\
-2x + 4z &= 10 \\
4x - 3y &= -2
\end{align*}
\]

Augmented Matrix:

\[
\begin{bmatrix}
4 & 2 & -3 & -1 \\
-2 & 0 & 4 & 10 \\
4 & -3 & 0 & -2
\end{bmatrix}
\]

5. Write the systems of equations that goes with the Augmented Matrix:

\[
\begin{align*}
2x + 4y &= 8 \\
-x + 5y + z &= 11 \\
4y + 2z &= 13
\end{align*}
\]

When solving systems of equations with Matrices, the goal is to get the Matrix into Row Echelon Form.

**Row Echelon Form**

Examples:

\[
\begin{bmatrix}
1 & 4 & 5 & 2 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

**Reduced Row Echelon Form**

Examples:

\[
\begin{bmatrix}
1 & 2 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

**Row Echelon Form:**

1. All rows consisting of entirely zeros are last row
2. The first non-zero entry of each row is 1.
3. The row below will have a first entry that starts in the column after the previous row.

**Reduced Row Echelon Form:**

The leading entry in each row is the only non-zero entry in its column. \( (\text{all #s below and above 1 is 0}) \)

***Row Echelon Form answers can vary, however there is only one Reduced Row Echelon answer for each matrix***
To get a Matrix into Row Echelon Form, we use Elementary Row Operations.

\[
\begin{bmatrix}
1 & 2 & -1 & 3 \\
3 & 7 & -5 & 14 \\
-2 & -1 & -3 & 8
\end{bmatrix}
\]

1. Perform the following operations:
   - \(3R_1 + R_2 \rightarrow R_2\)
   - \(2R_1 + R_3 \rightarrow R_3\)

\[
\begin{bmatrix}
1 & 2 & -1 & 3 \\
0 & 1 & -2 & 5 \\
0 & 3 & -5 & 14
\end{bmatrix}
\]

2. Perform the following operations:
   - \(-3R_2 + R_3 \rightarrow R_3\)

\[
\begin{bmatrix}
1 & 2 & -1 & 3 \\
0 & 1 & -2 & 5 \\
0 & 0 & 1 & -1
\end{bmatrix}
\]

Answers may vary!

On Calculator find the reduced row echelon form:

\[
\begin{bmatrix}
1 & 0 & 0 & 4 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & -1
\end{bmatrix}
\]

One Solution

Solving Systems using Gaussian Elimination:

\[
\begin{align*}
2x + 6y &= 16 \\
2x + 3y &= 7
\end{align*}
\]

Steps:

1. Write augmented matrix

\[
\begin{bmatrix}
2 & 6 & 16 \\
2 & 3 & 7
\end{bmatrix}
\]

2. Perform elementary row operations to get into row echelon form.

\[
\begin{bmatrix}
1 & 3 & 8 \\
2 & 3 & 7
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 3 & 8 \\
0 & -3 & -9
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 3 & 8 \\
0 & 1 & 3
\end{bmatrix}
\]

X + 3y = 8

Y = 3

X + 3(3) = 8

X = -1

\((-1, 3)\)
Solving Systems using **Gauss-Jordan Elimination**

\[-x + y = 4\]
\[2x - 4y = -34\]

Step 1: Write augmented matrix

\[
\begin{bmatrix}
-1 & 1 & 4 \\
2 & -4 & -34
\end{bmatrix}
\]

Step 2: Perform elementary row operations to get into reduced row echelon form.

\[
R_2 + R_1 \Rightarrow \begin{bmatrix}
1 & -3 & -30 \\
2 & -4 & -34
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 9 \\
0 & 1 & 13
\end{bmatrix}
\]

Steps 3: Identify each solution

\[(x, y) = (9, 13)\]

**On the Calculator:**

1. \[-x + 2y = 1.5\]
2. \[x - 3y = 5\]

\[
\begin{bmatrix}
-1 & 2 & 1.5 \\
2 & -4 & 3
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

No Solution: A zero row equal to a number

**Infinite Solutions:** An entire row of zeros